Realization Problems on Reachability Sequences

COCOON 2020

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August 30, 2020

Overview

1. The Reachability Realization Problem
2. Our Results
3. Approximation Algorithms
   - Notion of Bicriteria Approximation
   - Linear Program Randomized Rounding
   - Deterministic Sieving using Hurkens-Schrijver (DSHS)
4. Open Problems

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The Reachability Realization Problem

- **Reachability value** of a node in a digraph:
  Number of nodes reachable from the given node.
- **Reachability Sequence**:
  A sequence of all reachability values of nodes in the digraph.
- **Reachability Realization problem**:
  Is there a digraph with the given Reachability Sequence.
- We look at reachability realization for directed acyclic graphs (DAGs).
- Reminiscent of the Graph Realization problem on Degree Sequences [EG60, Hav55, Hak62].

Our results show an interesting interplay between the local property of degree and the global property of reachability.

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Our Results

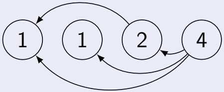
			Out-degree	
			Unbounded	Bounded
In-Degree	Unbounded	(DAGs)	Linear-time	$(O(\log n), O(\log n))$
	Bounded	(Trees)	$(O(\log n), O(\log n))$	$(O(\log n), O(\log n))$

Linear-time Algorithm for DAGs

Theorem (DAG reachability)

Given a reachability sequence $\{r_1, r_2, \ldots, r_n\}$ in non-decreasing order there exists a DAG that realizes it iff $r_i \leq i$ for all i.

Proof.



1. Only reach nodes with a strictly lower reachability value.
2. Only if: $r_i \leq i$ as at most $i - 1$ other nodes have lower reachability.
3. For all i, connect node i to the first $r_i - 1$ nodes.
4. Reachability is exactly r_i as we connect to all children of a node before connecting to a node.

□

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Notion of Bicriteria Approximation

- **δ-Degree consistency:**
  Graph G is δ-degree consistent with graph H if for all nodes i:

$$I_G(i) \in \Big[I_H(i), (1+\delta)\cdot I_H(i)\Big] \quad \text{and} \quad O_G(i) \in \Big[O_H(i), (1+\delta)\cdot O_H(i)\Big]$$

- **ρ-reachability consistency:**
  A tree G is ρ-reachability consistent to sequence r_i if for all nodes i:

$$r_i \leq 1 + \sum_{j\in C(i)} a_j \leq \rho \cdot r_i$$

  where a_i are the reachability labels in the approximate solution.

- G (ρ, δ)-**approximates** graph H if it is ρ-reachability consistent and δ-degree consistent with H.

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Linear Program Randomized Rounding

Theorem (LPRR)

Given a reachability sequence for a full k-ary tree, T, we can construct a DAG that is an $(O(\log n), O(\log n))$*-approximation to T in* $O(n^{\omega+\frac{1}{18}})$*-time.*

Let f_{ij} be the flow from node i to node j.

$$\begin{aligned} \min \quad & 1 \\ \text{s. t.} \quad & \sum_j f_{ji} = I(i) \;\; \forall i, && \text{In-degree requirement} \\ & \sum_j f_{ij} = O_G(i) \;\; \forall i, && \text{Out-degree requirement} \\ & r_i = 1 + \sum_j f_{ij} \cdot r_j \;\; \forall i, && \text{Reachability consistency} \\ & f_{ij} = 0 \;\; \forall i, j \text{ s.t. } r_i \leq r_j && \text{Acyclicity} \end{aligned}$$

Linear Program Randomized Rounding (Cont.)

- Round each edge ij to 1 w.p. f_{ij} independently $24 \ln n$ times.
- Expected in-degree value $\mu_{in} = 24 \ln n \cdot I(i)$.
- By Chernoff bound, $\Pr\left[\text{In-degree not in } (1 \pm \frac{1}{2})\mu_{in}\right] \leq \frac{2}{n^2}$.
- A similar argument applies for out-degrees and reachability values.
- Union bound $\implies \Pr\left[\text{Algorithm Failure}\right] \leq 3n \cdot \frac{2}{n^2} = o(1)$.

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The Hurkens-Schrijver t-set packing algorithm

- Given a collection of sets with each set of cardinality t, the t-set packing problem is to find the largest *disjoint* sub-collection.

- This has an $n^{O(t^3)}$-time $\frac{3}{t+3}$ approximation algorithm [FY14, HS89].

- Our algorithm, Deterministic Sieving using Hurkens-Schrijver (DSHS) has two phases, both of which solve $(k+1)$-set packing problems.

Theorem (Deterministic Sieving using Hurkens-Schrijver)

Given a reachability sequence for a full k-ary tree, T, we can construct a DAG that is an $(O(\log n), O(\log n))$-approximation to T in $n^{O(k^3)}$-time.

Deterministic Sieving using Hurkens-Schrijver (DSHS)

Proof Sketch.

1. *MatchChildren:* ensures that every node's out-degree is satisfied.
   - Universe consisting of V and P_t, all nodes that need children.
   - Collection: Pick a node $i \in P_t$ and $j_1, j_2, \ldots, j_k$ from V such that $r_i = 1 + r_{j_1} + r_{j_2} + \ldots + r_{j_k}$.

2. *MatchParent:* ensures that every node's in-degree is satisfied.
   - Universe consisting of all nodes V and C_t, the candidate nodes.
   - Collection: Pick a child node $i \in C_t$ and $j, j_1, j_2, \ldots, j_{k-1} \in V$ such that $r_j = 1 + r_i + r_{j_1} + r_{j_2} + \ldots + r_{j_{k-1}}$.

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Summary and Open Problems

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Open Problems:

- Derandomizing LPRR to reduce multi-edges in the solutions.
- Algorithms with good running time *and* simple solutions.
- If better approximation isn't possible, hardness of approximation.
- Graphs with cycles; our results are limited to acyclic graphs.

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